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Uniform Approximation of Completely Monotone Functions by Exponential Sums*

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1. INTRODUCTION

Let F be a completely monotone function on $[0, \infty]$, i.e.,

$$F \in C^\infty(0, \infty) \cap C[0, \infty],$$
$$(-1)^m F^{(m)}(t) \geqslant 0, \qquad 0 < t < \infty, \qquad m = 0, 1, \ldots .$$

The approximation of such functions by real exponential sums from

$$V_n(\mathbb{R}) = \{Y \in C^n(\mathbb{R}): [(D + \lambda_1) \cdots (D + \lambda_n)]Y \equiv 0 \quad \text{for some } \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}$$

has been studied by Braess [1] and by Kammler [4, 5]. In [4, Theorem 1] it is shown that there is always a best uniform approximation, Y, to F from $V_n(\mathbb{R})$ on an interval of the form $T = [a, b]$, $0 \leqslant a < b \leqslant \infty$, and that Y has the representation

$$Y(t) = a_1 e^{-\lambda_1 t} + \cdots + a_n e^{-\lambda_n t} \tag{1}$$

with the parameters satisfying the constraints

$$0 \leqslant \lambda_1 < \cdots < \lambda_n, \qquad a_i \geqslant 0, \quad i = 1, \ldots, n. \tag{2}$$

Braess [1, p. 230] has indicated that a completely monotone function can be interpolated at $2n$ distinct points by an exponential sum of the form (1)–(2). In the special case where the nodes are equally spaced Kammler [5, Theorem 1] has demonstrated that Prony's method (cf. [3, pp. 457–462]) can be used to construct the interpolating exponential sum.

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It is well known (cf. [9, p. 65]) that a best uniform approximation from $V_n(\mathbb{R})$ to an arbitrary function F on a finite set T may not exist. Nevertheless, when F is completely monotone, the existence and characterization theory of [4] which applies when we use the family of exponential sums $V_n(\mathbb{R})$ to approximate F on an interval $T = [a, b]$ using the uniform norm

$$\|F\| = \sup\{|F(t)|: t \in T\}$$

extends to the case where T is any closed subset of $[0, \infty)$ (and thus in particular to the case where T is discrete). In this paper we shall develop this extension. In so doing we shall make use of the following result from [4, Lemma 4].

LEMMA 1. *Let F be a completely monotone function on $[0, \infty]$ and let $Y \in V_n(\mathbb{R})$. If $F - Y$ has more than $2n$ nonnegative zeros, then $Y \equiv F$, and if $F - Y$ has exactly $2n$ nonnegative zeros $z_1 < \cdots < z_{2n}$ then Y has the representation* (1) — (2) *with $\lambda_1 > 0$, with $a_i > 0$ for each i, and with $F - Y$ positive on $(z_{2n}, +\infty)$.*

2. Alternation of Approximations

We say that the error curve $\epsilon = F - Y$ alternates m times on the closed set T if there exists $m + 1$ points of T, $t_1 < \cdots < t_{m+1}$, such that

$$|\epsilon(t_1)| = \|\epsilon\|$$

and

$$\epsilon(t_i) = -\epsilon(t_{i+1}), \qquad i = 1,..., m.$$

When the set T consists of the $2n + 1$ distinct points

$$t_1 < \cdots < t_{2n+1} \tag{3}$$

then ϵ alternates $2n$ times on the points (3) provided there is a constant d such that

$$F(t_i) = Y(t_i) - (-1)^i d, \qquad i = 1,..., 2n + 1. \tag{4}$$

If F is completely monotone on $[0, \infty]$, $Y \in V_n(\mathbb{R})$, and (4) holds, then from Lemma 1 we see that $d \geqslant 0$ and that Y has the form (1) — (2).

When the points (3) are equally spaced Meinardus' variant of Prony's method [5] can be used to produce an exponential sum Y and the corresponding constant d which satisfy (4). In this section we extend this result to the case where the points (3) are not equally spaced.

We begin with three preparatory lemmas.

LEMMA 2. *Let $Y \in V_n(\mathbb{R})$. Then Y has at most $n - 1$ real zeros unless $Y \equiv 0$.*

Proof. See Pólya and Szegö [8, p. 48, No. 75]. ∎

LEMMA 3. *Let $\{Y_\nu\}$ be a sequence of exponential sums from $V_n(\mathbb{R})$ such that each Y_ν has the representation*

$$Y_\nu(t) = a_{1\nu}e^{-\lambda_{1\nu}t} + \cdots + a_{n\nu}e^{-\lambda_{n\nu}t}$$

with the parameters satisfying (2), *and such that the sequence $\{Y_\nu(0)\}$ is bounded. Then some subsequence of $\{Y_\nu\}$ converges pointwise on* $(0, \infty)$ *to an exponential sum Y of the form* (1) — (2) *with the convergence being uniform on closed subsets of* $(0, \infty)$.

Proof. Since $\{Y_\nu(0)\}$ is bounded each of the sequences $\{a_{i\nu}\}$, $i = 1,..., n$ is bounded. Thus, after passing to a subsequence and reindexing, if necessary, we may assume that each $\{a_{i\nu}\}$ has limit $a_i \geqslant 0$ and that there is an l, $0 \leqslant l \leqslant n$, such that each $\{\lambda_{i\nu}\}$ has the finite limit $\lambda_i \geqslant 0$ for $i \leqslant l$ and the limit $+\infty$ for $i > l$. The exponential sum

$$Y(t) = \sum_{1 \leqslant i \leqslant l} a_i e^{-\lambda_i t}$$

(with the empty sum denoting the zero function) is the desired limit of $\{Y_\nu\}$. ∎

LEMMA 4. *Let F be a completely monotone function on* $[0, \infty]$ *and let*

$$t'_1 < \cdots < t'_{2n+1} \tag{5}$$

be $2n + 1$ points from $(0, \infty)$. *Assume there is an exponential sum Y' of the form*

$$Y'(t) = a'_1e^{-\lambda'_1 t} + \cdots + a'_n e^{-\lambda'_n t}$$

with the parameters satisfying (2) *such that $F - Y'$ alternates $2n$ times on the points* (5), *and let $d' = F(t'_1) - Y'(t'_1) \geqslant 0$. Let $k = 2,..., 2n$ be chosen and let $t' \in (t'_{k-1}, t'_{k+1})$ be selected. Then there is a unique exponential sum Y of the form* (1) — (2) *such that $F - Y$ alternates $2n$ times on the $2n + 1$ points formed from* (5) *by replacing t'_k with t'.*

Proof. We may assume that F is not itself an exponential sum of the form (1) — (2). Given $\mathbf{t} = (t_1, ..., t_{2n+1}) \in \mathbb{R}^{2n+1}$ with $0 < t_1 < \cdots < t_{2n+1}$ and $\mathbf{p} = (a_1, ..., a_n, \lambda_1, ..., \lambda_n, d) \in \mathbb{R}^{2n+1}$ we define

$$\mathbf{\Phi}(\mathbf{t}, \mathbf{p}) = (\phi_1(\mathbf{t}, \mathbf{p}), ..., \phi_{2n+1}(\mathbf{t}, \mathbf{p}))$$

where

$$\phi_i(\mathbf{t}, \mathbf{p}) = F(t_i) - \sum_{j=1}^{n} a_j e^{-\lambda_j t_i} + (-1)^i d, \qquad i = 1,..., 2n + 1.$$

The mapping $\mathbf{\Phi}$ is continuously differentiable with $\mathbf{\Phi}(\mathbf{t}', \mathbf{p}') = \mathbf{0}$ where the components $a'_1, ..., a'_n, \lambda'_1, ..., \lambda'_n, d'$ of $\mathbf{p}'$ are the parameters of the exponential sum Y' and the constant d', respectively, and where the components of $\mathbf{t}'$ are the points (5). By hypothesis, the parameters of Y' satisfy (2), and since we are assuming that F is not of the form (1) — (2), Lemma 1 shows that each component of $\mathbf{p}'$ is strictly positive. Consequently, since the $2n$ functions

$$e^{-\lambda'_i t}, \qquad te^{-\lambda'_i t}, \qquad i = 1,\dots, n,$$

form a Haar system [7, p. 177] it follows that the matrix

$$\mathbf{D}(\mathbf{t}, \mathbf{p}) = \left[\frac{\partial \phi_i(\mathbf{t}, \mathbf{p})}{\partial_{p_j}}\right]_{i,j=1}^{2n+1}$$

is nonsingular at $(\mathbf{t}', \mathbf{p}')$.

Using the implicit function theorem [2, p. 265], we infer the existence of α, β where $t'_{k-1} \leqslant \alpha < t'_k < \beta \leqslant t'_{k+1}$ and a continuous mapping $\mathbf{P}$ defined on

$$T_{\alpha\beta} = (t'_1,\dots, t'_{k-1}, t_k, t'_{k+1},\dots, t'_{2n+1})\colon \alpha < t_k < \beta\}$$

such that

$$\mathbf{P}(\mathbf{t}') = \mathbf{p}' \tag{6}$$

and

$$\mathbf{\Phi}(\mathbf{t}, \mathbf{P}(\mathbf{t})) = \mathbf{0}, \qquad \mathbf{t} \in T_{\alpha\beta}, \tag{7}$$

i.e., the statement of the lemma holds provided that t' is restricted to $(\alpha, \beta) \subseteq (t'_{k-1}, t'_{k+1})$.

To complete the proof we must show that it is possible to take $\alpha = t'_{k-1}$ and $\beta = t'_{k+1}$. Suppose that (6) — (7) hold and that $t'_k < \beta < t'_{k+1}$. Let $\{t_{k\nu}\}$ be a nondecreasing sequence from (t'_k, β) with limit β and for each $\nu = 1, 2,\dots$ let

$$\mathbf{t}_\nu = (t'_1,\dots, t'_{k-1}, t_{k\nu}, t'_{k+1},\dots, t'_{2n+1}),$$

let

$$(a_{1\nu},\dots, a_{n\nu}, \lambda_{1\nu},\dots, \lambda_{n\nu}, d_\nu) = \mathbf{P}(\mathbf{t}_\nu),$$

and let

$$Y_\nu(t) = a_{1\nu}e^{-\lambda_{1\nu}t} + \cdots + a_{n\nu}e^{-\lambda_{n\nu}t}.$$

By Lemma 1 we see that each component of $\mathbf{P}(\mathbf{t}_\nu)$ is nonnegative. Since $0 \leqslant Y_\nu(0) \leqslant F(0)$ for each $\nu = 1, 2,\dots$ the sequence $\{Y_\nu(0)\}$ is bounded and by Lemma 3 we can assume that $\{Y_\nu\}$ uniformly converges on $[t'_1, t'_{2n+1}]$ to an exponential sum Y of the form (1) — (2). It follows that $\{d_\nu\}$ converges to some $d \geqslant 0$ and that $F - Y$ alternates $2n$ times on the points $t'_1 < \cdots < t'_{k-1} < \beta < t'_{k+1} < \cdots < t'_{2n+1}$. By again applying the implicit function theorem at the point $(t'_1,\dots, t'_{k-1}, \beta, t'_{k+1},\dots, t'_{2n+1})$ we see that $\mathbf{P}$ can be continuously extended to a larger domain $T_{\alpha\beta'}$, with $\beta < \beta' \leqslant t'_{k+1}$. It follows that we may take $\beta = t'_{k+1}$. Analogously, we may take $\alpha = t'_{k-1}$.

Using Lemma 2 in conjunction with a standard zero counting argument, we infer that Y is unique. ∎

We can now prove the desired theorem.

THEOREM 1. *Let F be a completely monotone function on* $[0, \infty]$ *and let the* $2n + 1$ *nonnegative points* (3) *be given. Then there is a unique exponential sum Y of the form* (1) — (2) *and a constant* $d \geqslant 0$ *such that* (4) *holds.*

Proof. If the points (3) are equally spaced then the desired exponential sum can be obtained using Meinardus' variant of Prony's method.

Now suppose the points are not equally spaced. When $t_1 > 0$ we begin with the exponential sum $Y' \in V_n(\mathbb{R})$ such that $F - Y'$ alternates $2n$ times on the uniformly spaced points

$$t'_i = t_1 + (i - 1)(t_{2n+1} - t_1)/2n, \qquad i = 1,..., 2n + 1,$$

and, using Lemma 4, we successively adjust the position of the interior points $t'_2, ..., t'_{2n}$ to coincide with $t_2, ..., t_{2n}$.

To invoke the above lemma and hence the implicit function theorem t_1 must lie in the interior of $[0, \infty)$, and so we separately consider the case $t_1 = 0$. Using the result of the Theorem for the case $t_1 > 0$ together with Lemma 3, we can construct a sequence of exponential sums which uniformly converges on $[0, t_{2n+1}]$ to the desired exponential sum Y. ▮

By allowing two consecutive points of (3) to coalesce, we obtain the following interpolation result.

COROLLARY. *Let F be a completely monotone function on* $[0, \infty]$ *and let* $2n$ *distinct nonnegative points be selected. Then there is a unique exponential sum Y of the form* (1) — (2) *which interpolates F at these* $2n$ *points.*

Alternative arguments for the existence of the exponential sum interpolant of the preceding corollary are given in [6, p. 75], [1, p. 230], and when the nodes are equally spaced [5, Theorem 1].

3. EXISTENCE OF BEST APPROXIMATIONS

We now establish the following existence theorem.

THEOREM 2. *Let n be a positive integer, let T be a nonempty closed subset of* $[0, \infty)$, *and let F be a completely monotone function on* $[0, \infty]$ *with* $F(\infty) = 0$ *if* $\sup T = +\infty$. *Then there is a best uniform approximation, Y, to F on T from* $V_n(\mathbb{R})$, *and Y has the representation* (1) — (2). *This exponential sum is the unique best approximation to F from* $V_n(\mathbb{R})$ *except in the degenerate case where T contains fewer than* $2n$ *points. Furthermore, if T contains at least* $2n + 1$ *points then*

$Y' \in V_n(\mathbb{R})$ is the unique best approximation if and only if $Y' \equiv F$ or if $F - Y'$ alternates exactly $2n$ times on T.

Proof. In view of the Corollary to Theorem 1 it is sufficient to consider the case where $F \notin V_n(\mathbb{R})$ and where T contains at least $2n + 1$ distinct points.

Let the points $t_{11} < \cdots < t_{2n+1,1}$ be chosen from T and in accordance with Theorem 1 let the exponential sum $Y_1 \in V_n(\mathbb{R})$ and the corresponding constant $d_1 > 0$ be selected so that $F - Y_1$ alternates $2n$ times on these points with $F(t_{11}) - Y_1(t_{11}) = d_1 > 0$.

In view of Lemma 1 the function $F - Y_1$ has exactly $2n$ positive zeros $z_{11} < \cdots < z_{2n,1}$. We set $z_{01} = 0$, $z_{2n+1,1} = +\infty$ and we let t_{i2} be a point where $|F(t) - Y_1(t)|$ takes its maximum on the set $[z_{i-1,1}, z_{i1}] \cap T$, $i = 1,\ldots, 2n + 1$. Thus, by construction

$$(-1)^{i-1}[F(t_{i2}) - Y_1(t_{i2})] \geqslant d_1, \qquad i = 1,\ldots, 2n + 1.$$

By Theorem 1 there is a unique exponential sum Y_2 and a positive constant d_2 such that

$$(-1)^{i-1}[F(t_{i2}) - Y_2(t_{i2})] = d_2, \qquad i = 1,\ldots, 2n + 1,$$

and since $Y_2 - Y_1$ has at most $2n$ zeros unless $Y_2 \equiv Y_1$, it follows that $d_2 \geqslant d_1$.

Continuing in this fashion we generate a sequence of exponential sums $\{Y_\nu\}$ from $V_n(\mathbb{R})$ such that each error curve $F - Y_\nu$ alternates $2n$ times on the points $t_{1\nu} < \cdots < t_{2n+1,\nu}$ with $F(t_{1\nu}) - Y_\nu(t_{1\nu}) = d_\nu > 0$, $\nu = 1, 2,\ldots$. Since $\{d_\nu\}$ is nondecreasing and since

$$0 \leqslant F(t_{1\nu}) - Y_\nu(t_{1\nu}) = d_\nu \leqslant F(0), \qquad \nu = 1, 2,\ldots,$$

the sequence has a positive limit d. Since T is closed and since $F(\infty) = 0$ if $\sup T = +\infty$, we may assume after passing to a subsequence, if necessary, that $\{t_{i\nu}\}$ has limit $t_i \in T$, $i = 1,\ldots, 2n + 1$, where $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{2n+1}$. Moreover, since

$$F(t_{1\nu}) - d_\nu = Y_\nu(t_{1\nu}) \geqslant Y_\nu(t_{2\nu}) = F(t_{2\nu}) + d_\nu$$

we have

$$F(t_{1\nu}) - F(t_{2\nu}) \geqslant 2d\nu > d_1 > 0, \qquad \nu = 1, 2,\ldots,$$

and it follows that $t_1 < t_2$. By Lemma 3 we can assume that $\{Y_\nu\}$ uniformly converges on $[t_2, t_{2n+1}]$ to an exponential sum Y of the form (1) — (2) satisfying

$$F(t_i) = Y(t_i) - (-1)^i d, \qquad i = 2,\ldots, 2n + 1. \tag{8}$$

Hence, $t_1 < t_2 < \cdots < t_{2n+1}$, and so $F - Y$ has at least $2n - 1$ distinct positive zeros. Since $F \notin V_n(\mathbb{R})$ it follows from Lemma 1 that each coefficient of Y must be positive. Thus, (8) also holds for $i = 1$, and $\{Y_\nu\}$ uniformly converges to Y on $[0, \infty)$.

Finally,

$$\begin{aligned} \|F - Y\| &= \lim_{\nu\to\infty} \|F - Y_\nu\| \\ &= \max_{1\leqslant i\leqslant 2n+1} \lim_{\nu\to\infty} |F(t_{i,\nu+1}) - Y_\nu(t_{i,\nu+1})| \\ &= \max_{1\leqslant i\leqslant 2n+1} |F(t_i) - Y(t_i)| = d. \end{aligned}$$

In conjunction with (8) this shows that $F - Y$ alternates exactly $2n$ times on T so that Y is indeed the unique best approximation to F from $V_n(\mathbb{R})$. ∎

4. A Numerical Example

Using the Remez simultaneous exchange method (cf. [7, pp. 105–116]) described in the above proof of Theorem 2, we have computed the best uniform approximation to $F(t) = (1 + t)^{-1}$ from $V_3(\mathbb{R})$ on the set $T = \{0, 1, 2,...\}$.

TABLE I

Uniform Approximation of $F(t) = (1 + t)^{-1}$ on the Nonnegative Integers by $Y_\nu(t) = \sum_{1\leqslant i\leqslant 3} a_{i\nu} \exp(-\lambda_{i\nu} t)$

ν	$a_{i\nu}$	$\lambda_{i\nu}$	d_ν	Extremal points
1	0.25930576	0.11037788	3.4897×10^{-5}	0, 1, 2, 3, 4, 5, 30
	0.43940068	0.64615087		
	0.30125865	2.08214161		
2	0.16331841	0.05422944	1.5900×10^{-4}	0, 1, 2, 3, 4, 14, 70
	0.46032659	0.47763783		
	0.37619598	1.83667799		
3	0.09328788	0.02769839	6.6375×10^{-3}	0, 1, 2, 3, 8, 28, 144
	0.43950729	0.33515819		
	0.46654106	1.58570596		
4	0.06015817	0.01797400	2.3756×10^{-3}	0, 1, 2, 4, 12, 44, 221
	0.37860905	0.23751079		
	0.55885716	1.35107679		
5	0.06579775	0.02102968	3.8738×10^{-3}	0, 1, 2, 5, 13, 39, 183
	0.35403951	0.23180982		
	0.57628889	1.28986014		
6	0.06623901	0.02116341	4.0572×10^{-3}	0, 1, 2, 5, 13, 39, 182
	0.35363119	0.23317412		
	0.57607264	1.28655114		
7	0.06623759	0.02116278	4.0573×10^{-3}	0, 1, 2, 5, 13, 39, 182
	0.35362995	0.23317133		
	0.57607519	1.28654525		

We used Meinardus' variant of Prony's method to compute the initial approximation from $V_3(\mathbb{R})$, Y_1, so that $F - Y_1$ alternates six times on the points 0, 1, 2, 3, 4, 5, 6 with $F(0) - Y_1(0) = d_1$. Using Newton's method, we then computed $Y_\nu \in V_3(\mathbb{R})$ and $d_\nu > 0$ so that $F - Y_\nu$ alternates six times on the extremal points, $t_{1\nu} < \cdots < t_{7\nu}$, of $F - Y_{\nu-1}$ with $F(t_{1\nu}) - Y_\nu(t_{1\nu}) = d_\nu$, $\nu = 2,..., 7$. Since the extremal points of $F - Y_6$ and $F - Y_7$ are identical Y_7 is the best approximation. Our results (with the parameters truncated) are displayed in Table I.

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